

# Bilayer Graphene as a platform for Bosonic Symmetry Protected Topological States

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Bosonic symmetry protected topological (BSPT) states, *i.e.* bosonic analogue of topological insulators, have attracted enormous theoretical interests and efforts in the last few years. Although the BSPT states have been successfully classified with various approaches, there has been no successful experimental realization of BSPT states yet in two and higher dimensions. In this paper, we propose that the two dimensional BSPT state with  $U(1) \times U(1)$  symmetry can be realized in a bilayer graphene under an out-of-plane magnetic field, where the two  $U(1)$  symmetries stand for the total spin  $S^z$  and total charge conservation respectively. The Coulomb interaction plays a central role in this proposal: 1. it gaps out all the fermions at the boundary of the system, hence the remaining symmetry protected gapless boundary states only have bosonic charge and spin degrees of freedom; 2. based on the conclusion above, we propose that the bulk quantum phase transition between the BSPT and trivial phase, which can be driven by a competition between the out-of-plane magnetic field and electric field, under strong interaction can become a “bosonic phase transition”, *i.e.* only bosonic modes close their gap at the transition. This transition is fundamentally different from all the well-known topological-trivial transition in the free fermion topological insulators. The latter statement is supported by recent determinant quantum Monte carlo simulation on a similar sign-problem-free model on a bilayer honeycomb lattice. Various experimental consequences will be predicted.

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A symmetry protected topological (SPT) state, first defined in Ref. 1, 2, refers to the ground state of a local quantum many-body Hamiltonian whose bulk is gapped and nondegenerate, but its boundary remains either gapless or degenerate as long as the entire system including the boundary preserves certain symmetries. The fermionic SPT states include the familiar quantum spin Hall (QSH) insulator [3, 4], the three-dimensional (3d) topological insulator (TI) [5–7], and topological superconductors. Noninteracting fermionic SPT states have been fully classified and understood [8–10]. Unlike fermionic systems, bosonic SPT (BSPT) states all need strong interaction to overcome the tendency to form some ordered states such as Bose-Einstein condensate. The simplest and most well-known BSPT state is the 1d Haldane phase that can be realized as a simple Heisenberg model of a spin-1 chain [11, 12]. However, the higher dimensional generalizations of BSPT states, though have attracted enormous attentions and theoretical efforts in the last few years, still lack a feasible experimental realization in condensed matter systems, except for a proposal of realization with ultracold atoms [13]. Especially, the exactly soluble parent Hamiltonians constructed in Ref. 1, 2 in dimensions higher than one all involve high order multiple spin interactions, which are unlikely to exist in realistic materials; while so far different approaches of classifying and characterizing BSPT states [1, 2, 14–19] all rely on mathematical or effective field theory description, which shed little light on how to construct a realistic lattice model for BSPT states.

In the current paper, we hope to bridge the gap between theoretical studies and experimental realizations of the BSPT states. We propose that the bilayer graphene under an out-of-plane magnetic field provides a platform of realizing and probing the 2d BSPT state with  $U(1)_s \times U(1)_c$  symmetry, where  $U(1)_s$  and  $U(1)_c$  correspond to the total spin- $S^z$  and total electric charge conservation respectively. Based on the formalism developed in Ref. 15, 16, this state has a  $\mathbb{Z}$  classification, *i.e.* there are infinite classes of nontrivial 2d BSPT states under these symmetries, which is identical to the classification of the QSH state with  $S^z$  conservation. These BSPT states can either be described by the Chern-Simon field theory [15]

$$\mathcal{S} = \int d^2x d\tau \frac{ik}{2\pi} K_{IJ} a^I \wedge da^J, \quad (1)$$

where  $K_{IJ} = \sigma^x$ , and the flux of  $U(1)$  gauge fields  $a_\mu^1$  and  $a_\mu^2$  carry spin and charge respectively; or by the following nonlinear sigma model (NLSM) with a four component vector  $\mathbf{n} = (n_1, n_2, n_3, n_4)$  with unit length [16, 20]:

$$\mathcal{S} = \int d^2x d\tau \frac{1}{g} (\partial_\mu \mathbf{n})^2 + \frac{i\Theta}{\Omega_3} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c \partial_\tau n^d, \quad (2)$$

where  $\Omega_3$  is the volume of a 3d sphere with unit radius. In Eq. 2, the BSPT phases correspond to the strongly interacting fixed point  $g \rightarrow \infty$ , and  $\Theta \rightarrow 2k\pi$  with  $k$  being a nonzero integer, while the trivial phase corresponds to the fixed point  $\Theta \rightarrow 0$ . The latter description also gives

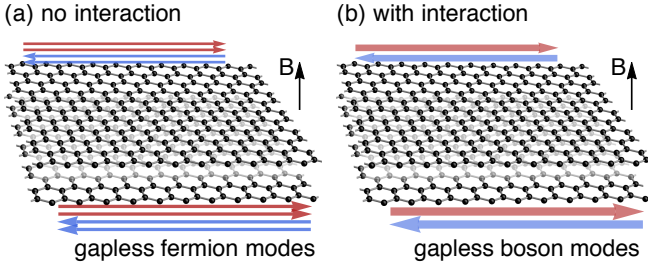


FIG. 1: Schematic setup of bilayer graphene in the presence of out-of-plane magnetic field. (a) If the interacting effect is not taken into account, the boundary will host two channels of fermionic edge states with central charge  $c = 2$ . (b) Including the Coulomb interaction, there is only one channel of bosonic edge state with  $c = 1$ .

us a field theory of the quantum phase transition between different BSPT phases: this transition is driven by tuning  $\Theta$ , and the critical point is at  $\Theta = (2k + 1)\pi$ . A similar phase diagram and renormalization group flow for NLSMs in one lower dimension was studied thoroughly in Ref. 21, 22.

For a bilayer graphene, once a strong enough out-of-plane magnetic field along  $\hat{z}$  direction is turned on, the bulk is driven into a “quantum spin Hall insulator” (it is also called the ferromagnetic quantum Hall state, since the bulk is fully spin polarized), with two channels of counter-propagating spin-filtered helical edge states [23–25]. We will demonstrate that, the bilayer graphene (as illustrated in Fig. 1), though built with electrons, under an out-of-plane magnetic field will mimic the behaviors of BSPT states in the following sense:

1. the Coulomb interaction, which is expected to play an important role in this system, will gap out all the fermions at the boundary, while the bosonic charge and spin degrees of freedom will survive at low energy due to the two  $U(1)$  symmetries of the system (Fig. 1b);
2. Using the Chalker-Coddington picture [26], the bulk quantum phase transition between phases with different topological nature can be described by percolation of

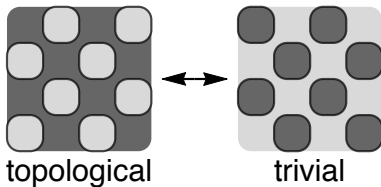


FIG. 2: Illustration of the Chalker-Coddington network. The darker (lighter) region corresponds to the topological (trivial) phase. The topological-trivial transition can be realized as a percolation transition. At the critical point, the bosonic boundary modes proliferate in the bulk along the network of interfaces.

domains and network of interface/boundary states (see Fig. 2). Because the boundary only has gapless bosonic modes, we expect that the bulk quantum phase transition between the BSPT state and the trivial insulator, which can be driven by the competition between the out-of-plane magnetic field and electric field in our system, also only has gapless bosonic modes but no gapless fermion modes under strong interaction. This is a qualitatively different situation from the well-known topological-trivial transition in free fermion systems, which always involves fermion gap closing in the bulk, such as the plateau transition between integer quantum Hall states. This statement above is supported by recent numerical studies of a similar model on the bilayer honeycomb lattice [27, 28].

3. Since the fermions never appear in any low energy physics of this system, fermions can be safely ignored, or “integrated out” from the low energy field theory, the resulting effective field theory will be purely bosonic, and it takes exactly the form as Eq. 2.

We start with two channels of helical edge states at one boundary of a bilayer graphene, described by the Hamiltonian

$$H_0 = \int dx \sum_{l=1}^2 \psi_{l,L} i v \partial_x \psi_{l,L} - \psi_{l,R} i v \partial_x \psi_{l,R}, \quad (3)$$

where  $l = 1, 2$  stand for the two channels of edge states,  $L, R$  stand for the left and right moving fermions respectively, which also correspond to electrons with spin-up and down, and  $v$  is the velocity of the edge states [57]. Note that in our convention  $v$  has the dimension of energy. The existence of the edge states has been observed experimentally [25]. When the Coulomb interaction is ignored, the boundary is a free fermion conformal field theory (CFT) with central charge  $c = 2$ . The edge state wave function is localized at the boundary, but it extends into the bulk with a localization length at the order of the magnetic length  $l_B$ , which is tunable by changing the magnetic field.

Without interaction, the free fermion edge states can be bosonized into two flavors of free bosons:

$$H_0 = \int dx \sum_{l=1}^2 \frac{v}{2K} (\partial_x \theta_l)^2 + \frac{vK}{2} (\partial_x \phi_l)^2, \quad (4)$$

where  $[\theta_l(x), \partial_{x'} \phi_{l'}(x')] = i\delta(x - x')\delta_{ll'}$ , and  $\psi_{l,L/R} \sim e^{i\theta_l \pm i\pi\phi_l}$ . For free 1d fermions without interaction, the Luttinger parameter  $K = \pi$ .

Under the Coulomb interaction  $H_{int}$ , the Luttinger parameter will be renormalized:

$$H_{int} = \int dx \sum_{l=1}^2 \frac{v\delta K}{2} (\partial_x \phi_l)^2 + v\delta K' \nabla_x \phi_1 \nabla_x \phi_2 + H_v, \quad (5)$$

where  $\delta K$  and  $\delta K'$  come from intralayer and interlayer interactions respectively.  $H_v$  corresponds to the anharmonic vertex term, and will play a central role here [58]:

$$H_v \sim \alpha \cos(2\pi\phi_1 - 2\pi\phi_2). \quad (6)$$

Here we have assumed that the long range Coulomb interaction is screened by the boundary states to a short range interaction, due to the finite density of states at the boundary. Physically  $H_v$  describes the backscattering between two channels of edge states:  $H_v \sim \psi_{1,L}^\dagger \psi_{1,R} \psi_{2,R}^\dagger \psi_{2,L}$ , which can be induced by the Coulomb interaction. The vertex operator  $H_v$  is relevant, as long as  $\delta K > \delta K'$ , or equivalently, the intralayer interaction is stronger than the interlayer interaction. This condition is naturally satisfied because in general there is a spatial separation between the two channels of edge states in the  $2d$  plane, which can be shown by explicitly solving the Schrodinger equation of the system with a boundary [24], or at the interface between the QSH insulator and the trivial insulator induced by an electric field along  $\hat{z}$  [29, 30].

A relevant  $H_v$  will gap out bosonic modes  $\theta_- = \theta_1 - \theta_2$  and  $\phi_- = (\phi_1 - \phi_2)/2$ , as well as all the fermions at the boundary. At the boundary, the bosonic modes  $(\phi_1 + \phi_2)/2 = \phi$  and  $\theta_1 + \theta_2 = \theta$  must remain gapless, because  $\theta$  transforms under symmetry  $U(1)_c$ , while  $\phi$  transforms under  $U(1)_s$ , *i.e.* the boundary has symmetry protected gapless bosonic modes. A naive estimate of the fermion gap will exceed the electron gap in the bulk, and we expect the boundary single particle gap to be comparable with the bulk fermion gap, which was estimated in Ref. 23.

The effective field theory that describes the canonical conjugate modes  $\phi$  and  $\theta$  reads

$$\tilde{H} = \int dx \frac{\tilde{v}}{2\tilde{K}} (\partial_x \theta)^2 + \frac{\tilde{v}\tilde{K}}{2} (\partial_x \phi)^2. \quad (7)$$

Hence interaction reduces the central charge of the system from  $c = 2$  to  $c = 1$ . Because  $\theta$  and  $\phi$  transform nontrivially under  $U(1)_c$  and  $U(1)_s$  symmetries respectively, there is no anharmonic vertex operators allowed by symmetry in Eq. 7. Because  $\theta$  and  $\phi$  are “dual” to each other, a unit soliton of  $\phi$  at the  $1d$  boundary carries charge- $2e$  (as illustrated in Fig. 3), and a unit soliton of  $\theta$  carries spin  $S^z = 1$ . The gaplessness of the boundary state is protected only by the  $U(1)_c \times U(1)_s$  symmetry, namely even if the translation symmetry of the boundary is broken by disorder (which is inevitable in the real system), as long as the disorder also preserves the  $U(1)_c \times U(1)_s$  symmetry, the boundary must still remain gapless. The edge state in our system is also very different from the cases studied in Ref. 31, 32, since in those systems the states localized at the domain wall is unstable against disorder.

Here we note that although the bosonization of the edge states of bilayer graphene under magnetic field was also studied in Ref. 29, 30, in these works only the spin symmetry was considered in the bosonization, and the conclusion of Ref. 29, 30 was that the system is effectively a  $1d$  spin model. Here we stress that, both the  $U(1)_s$  and  $U(1)_c$  symmetries are crucial to define the



FIG. 3: A unit soliton of  $\phi$  at the  $1d$  boundary corresponds to the winding of the spin XY order by  $2\pi$ , which will carry charge- $2e$  due to the topological effect.

BSPT state, *i.e.* if either of the  $U(1)$  symmetries is broken (for example if the bulk forms a canted antiferromagnetic order), the system will become a trivial state. And with both  $U(1)$  symmetries in our system, the boundary theory Eq. 7 must remain gapless, and it can never be realized as a  $1d$  system, *i.e.* it can only be realized as the boundary of a  $2d$  system, which is the very essential property of all the SPT states.

Assuming  $H_v$  already gaps out all the other modes, using the remaining fields  $\theta$  and  $\phi$ , one can define the four component vector  $\mathbf{n}$  in Eq. 2:

$$\begin{aligned} n_1 + in_2 &\sim \epsilon_{\alpha\beta} c_{1,\alpha} c_{2,\beta} \sim e^{i\theta + i2\pi\phi_-} - e^{i\theta - i2\pi\phi_-} \sim e^{i\theta}, \\ n_3 + in_4 &\sim \sum_l (-1)^l c_l^\dagger \sigma^+ c_l \sim e^{i2\pi\phi_1} - e^{i2\pi\phi_2} \sim e^{i2\pi\phi}. \end{aligned} \quad (8)$$

$n_1 + in_2$  corresponds to an interlayer spin-singlet Cooper pair, while  $n_3$  and  $n_4$  correspond to inplane antiferromagnetic order. All components of the vector  $\mathbf{n}$  have power-law correlation at the boundary, and their scaling dimensions are

$$\Delta[\epsilon_{\alpha\beta} c_{1,\alpha} c_{2,\beta}] = \frac{\tilde{K}}{4\pi}, \quad \Delta[\sum_l (-1)^l c_l^\dagger \sigma^+ c_l] = \frac{\pi}{\tilde{K}}. \quad (9)$$

Integrating out the gapped fermions, a  $(1+1)d$  NLSM for  $\mathbf{n}$  with a Wess-Zumino-Witten term at level  $k = 1$  is generated, whose infrared fixed point is a CFT with central charge  $c = 1$  [33, 34] (for more details please refer to the analysis in Ref. 35).

Knowing the effective field theory at the boundary is the  $(1+1)d$  NLSM for  $\mathbf{n}$  with a Wess-Zumino-Witten term at level  $k = 1$ , the bulk theory can be constructed with the Chalker-Coddington network model [26], and as was shown in Ref. 14, 36, the bulk theory obtained by this construction is precisely Eq. 2 with  $\Theta = 2\pi$ . The physical meaning of this topological  $\Theta$ -term is that, a vortex of  $(n_1, n_2)$ , *i.e.* a vortex of the superconductor order parameter which carries magnetic flux  $\frac{hc}{2e}$ , would carry spin  $S^z = 1$ , which is perfectly consistent with the physics of the bilayer QSH state.

Let us also stress that, for a single channel of QSH insulator, its boundary *cannot* be driven into a state with gapped fermions but gapless bosonic modes, as long as the  $U(1)_c$  and time-reversal symmetry of the QSH insulator are preserved [37, 38]. The mapping between

fermionic QSH insulator and BSPT is only valid for two copies of QSH insulators (which mathematically is equivalent to four copies of  $p \pm ip$  topological superconductors), as was shown in Ref. 39.

A competition between the electric field and magnetic field along  $\hat{z}$  direction will drive a quantum phase transition between the SPT and the trivial state. According to the Chalker-Coddington network picture, since the boundary state only has gapless bosonic modes, but no gapless fermion modes, we expect the bulk quantum phase transition also only has gapless boson modes under strong interaction, and in the field theory Eq. 2 this transition occurs when tuning  $\Theta$  to precisely  $\pi$ . Although directly analyzing the bulk field theory at  $\Theta = \pi$  is difficult, recent unbiased determinant quantum Monte Carlo simulation on a similar bilayer honeycomb lattice interacting fermion model confirms that this purely bosonic topological-trivial quantum phase transition can indeed happen [27, 28], which is fundamentally different from the ordinary topological-trivial transition in any free fermion system. Of course, in real experimental systems, there could be intermediate phases between the BSPT phase and the trivial phase, but we should still point out the highly interesting possibility of a direct second order “bosonic” transition like the one found in Ref. 27, 28.

#### *Experimental Implications*

The most obvious prediction of our theory is that in a Bernal-stacked graphene bilayer in the spin polarized phase [25], the gapless boundary modes are bosonic rather than fermionic. The low energy charge carriers on the edge are the Cooper pair  $\epsilon_{\alpha\beta}c_{1,\alpha}c_{2,\beta}$ , with charge  $2e$ , potentially allowing detection via shot noise measurements, which have previously been used to probe fractional charges in quantum Hall edge states [40–43]. Shot noise measurements could be performed in bilayer graphene by introducing a quantum point contact, either using electrostatic gates or a nanoconstriction, to introduce edge-to-edge backscattering and a finite transmission probability [43]. Alternatively, tunneling measurements could probe the bosonic nature of the edge carriers. The quasi-long range order of the Cooper-pair order parameter along the  $1d$  boundary implies a single-particle gap; much like in a superconductor, tunneling from a normal metal electrode or tip should show a hard gap, persisting until the fermion edge gap, despite ballistic, dissipationless in-plane resistance.

Similar effects might also be visible in twisted graphene bilayers, where, at large twist angle, the electronic structure is described by monolayers[44] with interlayer tunneling suppressed but interlayer Coulomb interactions largely preserved. In this geometry, putting both layers in the spin polarized  $\nu = 0$  phase[45] again realizes the two channel QSH state described above. Notably, gates can be used to control the ratio between inter- and intra-channel interactions, allowing exploration both of the limits described here ( $\delta K > \delta K'$  or  $\delta K' > \delta K$ ) as

well as the symmetric limit,  $\delta K \sim \delta K'$ , which will be the subject of future work.

If a direct second order quantum phase transition between the BSPT and trivial phase found in Ref. 27, 28 indeed happens in the real system, then at the transition, which corresponds to a  $(2 + 1)d$  CFT, the bulk conductivity should be a universal value  $\sigma = De^2/h$ , where  $D$  is an order-1 universal constant [46, 47]. Moreover the transition should be accompanied by a closing of the spin gap, with observable consequences for spin susceptibility as well as thermal transport measurements.

*Other experimental systems* - We stress that the potential material realization of BSPT states is not limited to bilayer graphene under magnetic fields, but is rather generic to two-channel helical states. The so-called topological mirror insulators (TMIs) can provide another platform to search for the BSPT phases. In a TMI, the  $U(1)_s$  in the spin channel is replaced by the mirror symmetry about  $\hat{z}$  axis, and as was shown in Ref. 48, an integer topological number, known as the mirror Chern number can be defined, leading to a  $\mathbb{Z}$  classification of a TMI at the free fermion level.  $2d$  TMIs with mirror Chern number  $\pm 2$  should similarly host two-channel helical edge modes and a bosonic SPT phase in the presence of interactions.  $2d$  TMIs can be engineered in thin films of  $3d$  TMIs [49–51], with the growth direction perpendicular to the mirror plane. This strategy has been applied to SnTe thin films [52] and rocksalt IV-VI semiconductors XY (X=Ge, Sn, Pb and Y= S, Se, Te) monolayers [53], as well as TlSe and TlS monolayers [54]. In particular,  $2d$  TMIs with mirror Chern number  $\pm 2$  have been identified in all of the above proposed materials. Since for a  $2d$  material the mirror symmetry about  $\hat{z}$  axis can be viewed as an internal  $Z_2$  symmetry of the system, according to Ref. 39, strong interaction can drive these  $2d$  TMIs into a BSPT with the  $Z_2$  symmetry, which is equivalent to the so called Levin-Gu model [55] and the CZX model [56].

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